Analysis of a Spectral Updating for the Method of Moving Asymptotes

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1. Abstract
The Method of Moving Asymptotes (MMA) is very popular within the structural optimization community and applies for inequality constrained nonlinear programming problems with simple bounds. In its more recent version (Svanberg, 2002), MMA was merged into the Conservative, Convex and Separable Approximation (CCSA) class of algorithms, which are globally convergent.

In this work a modified version of the MMA is proposed, based on the spectral parameter. The idea is to include the second-order information provided by this parameter into the model functions that define the rational approximations to the objective function and the nonlinear constraints, at the expense of additional gradient evaluations per inner iteration.

The spectral parameter with a safeguarding scheme plays an essential role in keeping the generated sequence conveniently conservative with respect to the original functions, preserving the global convergence property. As far as the objective function, conservative approximations ensure monotonically decreasing values, whereas for the constraints, feasibility for an augmented problem is guaranteed. Strict convexity and separability of the model functions are kept so that the subproblems have a unique solution and the large scale scenario may be addressed. Preliminary experiments indicate that the idea is promising, in the sense that the cost-benefit of computing the spectral parameter is worth for reducing the total effort of the algorithm as compared with the original version.

2. Keywords: constrained optimization, method of moving asymptotes, spectral parameter, large scale problems, global convergence.

3. Introduction
The Conservative Convex Separable Approximations (CCSA) class of methods, proposed by Svanberg in [8], aims to solve the following optimization problem:

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \\
& \quad x_j^{\min} \leq x_j \leq x_j^{\max}, \quad j = 1, \ldots, n,
\end{align*}
\]

(1)

where \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \) is the vector of the variables, \( x_j^{\min} \) and \( x_j^{\max} \) are given real numbers such that \( x_j^{\min} < x_j^{\max} \) for each \( j \) and \( f_0, f_1, \ldots, f_m \) are real-valued typically twice continuously differentiable functions.

Following Svanberg’s approach, artificial variables \( y = (y_1, \ldots, y_m)^T \) and \( z \in \mathbb{R} \) are introduced, so that the following enlarged problem is addressed:

\[
\begin{align*}
\text{minimize} & \quad f_0(x) + a_0 z + \sum_{i=1}^m (c_i y_i + \frac{1}{2} d_i y_i^2) \\
\text{subject to} & \quad f_i(x) - a_i z - y_i \leq 0, \\
& \quad x_j^{\min} \leq x_j \leq x_j^{\max}, \quad j = 1, \ldots, n, \\
& \quad y_i \geq 0, \quad i = 1, \ldots, m, \\
& \quad z \geq 0,
\end{align*}
\]

(2)

where \( a_0, a_i, c_i \) and \( d_i \) are real numbers such that \( a_0 > 0, a_i \geq 0, c_i \geq 0, d_i \geq 0 \) and \( c_i + d_i > 0 \) for \( i = 1, \ldots, m \). Moreover, \( a_i c_i > a_0 \) for all \( i \) such that \( a_i > 0 \). The constants \( c_i \) must be chosen large enough so that the variables \( y_i \) are zero at the optimal solution, in case the original problem has a nonempty feasible set. Problem (2) always has feasible points and at least an optimal solution, even if problem (1) has an empty feasible set; further, every local solution of problem (2) satisfies the Karush-Kuhn-Tucker conditions, since the feasible set of problem (2) is qualified in the sense that it naturally fulfills a regularity condition.
A CCSA method for solving problem 2 performs \( k \) outer and \( \ell \) inner iterations. The indices \((k, \ell)\) are used to denote the \( k \)-th inner iteration within the \( \ell \)-th outer iteration.

To start, it is necessary to choose \( x^{(1)} \in X = \{x \in \mathbb{R}^n; x_j^{\text{min}} \leq x_j \leq x_j^{\text{max}}, j = 1, \ldots, n\} \), and then to compute \( y^{(1)} \in z^{(1)} \), obtaining an initial feasible estimate \((x^{(1)}, y^{(1)}, z^{(1)})\) for problem (2).

Given \((x^{(k)}, y^{(k)}, z^{(k)})\), a subproblem is generated and solved. This subproblem is obtained from (2), replacing \( x \) by a convex subset \( X^{(k)} \), and the functions \( f_i(x), i = 0, 1, \ldots, m \) by separable strictly convex functions \( g_i^{(k, \ell)} \):

\[
\begin{align*}
\text{minimize} & \quad g_0^{(k, \ell)}(x) + a_0z + \sum_{i=1}^m (c_i y_i + \frac{1}{2} d_i y_i^2) \\
\text{subject to} & \quad g_i^{(k, \ell)}(x) - a_i z - y_i \leq 0, \quad \forall i = 1, \ldots, m, \\
& \quad x \in X^{(k)}, \quad y \geq 0, \quad z \geq 0.
\end{align*}
\]

for \( k \in \{1, 2, 3, \ldots \} \) and \( l \in \{0, 1, 2, \ldots \} \), where the set \( X^{(k)} \) is given by:

\( X^{(k)} = \{x \in X \mid x_j \in [x_j^{(k)} - 0.9\sigma_j^{(k)}, x_j^{(k)} + 0.9\sigma_j^{(k)}], \; j = 1, \ldots, n\} \).

The vector \( \sigma^{(k)} = (\sigma_1^{(k)}, \ldots, \sigma_n^{(k)})^T \) contains strictly positive parameters that are initially set and updated as stated later on in this manuscript.

The approximating functions \( g_i^{(k, \ell)} \) in the CCSA subproblem are chosen as:

\[
g_i^{(k, \ell)}(x) = v_i(x, x^{(k)}, \sigma^{(k)}) + \rho_i^{(k, \ell)} w_i(x, x^{(k)}, \sigma^{(k)}),
\]

where \( v_i(x, x^{(k)}, \sigma) \equiv v_i^{(k)}(x) \) and \( w_i(x, x^{(k)}, \sigma) \equiv w_i^{(k)}(x) \) are real-valued functions. To ensure that \( g_i^{(k, \ell)} \) possess suitable properties, some conditions must be satisfied, like continuity of the functions \( v_i^{(k)} \) and \( w_i^{(k)} \) and of their second-order derivatives with respect to \( x \) in an enlarged set that contains \( X \).

The parameters \( \rho_i^{(k, \ell)} \) are strictly positive, and within an outer iteration \( k \), the only difference between two inner iterations are the values of these parameters.

At the \( \ell \)-th inner iteration, if the conservativity condition holds at \( \hat{x}^{(k, \ell)} \) (solution of (3)):

\[
g_i^{(k, \ell)}(\hat{x}^{(k, \ell)}) \geq f_i(\hat{x}^{(k, \ell)}), \quad \forall i \in \{0, 1, \ldots, m\},
\]

then \((x^{(k+1)}, y^{(k+1)}, z^{(k+1)}) = (\hat{x}^{(k, \ell)}, \hat{y}^{(k, \ell)}, \hat{z}^{(k, \ell)})\), and the \( k \)-th outer iteration is completed. Otherwise, if \( g_i^{(k, \ell)}(\hat{x}^{(k, \ell)}) < f_i(\hat{x}^{(k, \ell)}) \) for at least an index \( i \in \{0, 1, \ldots, m\} \), the model \( g_i^{(k, \ell)}(x) \equiv g_i^{(k+1)}(x) \) for \( i \) such that the conservativity is fulfilled, and for the indices for which conservativity does not hold, the model is modified so that conservativity might be reached at the next inner iteration.

It is worth mentioning that conservativity is demanded for both the objective function and the constraints, producing strict reduction of the objective function value and feasible iterates, respectively.

The functions \( g_i^{(k, \ell)} \) are first-order approximations to the original functions \( f_i \) at the current estimate, that is, conditions \( g_i^{(k, \ell)}(x^{(k)}) = f_i(x^{(k)}) \) and \( \nabla g_i^{(k, \ell)}(x^{(k)}) = \nabla f_i(x^{(k)}) \) must hold. Besides, all the approximating functions \( g_i^{(k, \ell)} \) are strictly convex, so that every subproblem has a single global optimum. Another condition that must be satisfied by the approximating functions is the separability, that is, \( g_i^{(k, \ell)}(x) = g_{i0}^{(k, \ell)} + \sum_{j=1}^n g_{ij}^{(k, \ell)}(x_j) \). Such property is crucial in practice, because the Hessian matrices of the approximations are diagonal ones, allowing to address large-scale problems.

In the Method of Moving Asymptotes (MMA), a particular instance of the CCSA class, the approximating functions are given by:

\[
g_i^{(k, \ell)}(x) = \sum_{j=1}^n \left( \frac{\rho_{ij}^{(k, \ell)} x_j - l_j^{(k)}}{u_j^{(k)} - x_j} + \frac{q_{ij}^{(k, \ell)} x_j - l_j^{(k)}}{u_j^{(k)} - x_j} \right) + v_i^{(k, \ell)},
\]

where the poles of the moving asymptotes \( l_j^{(k)} \) and \( u_j^{(k)} \) are:

\[
l_j^{(k)} = x_j^{(k)} - \sigma_j^{(k)} \quad \text{and} \quad u_j^{(k)} = x_j^{(k)} + \sigma_j^{(k)},
\]
and the coefficients $p_i^{(k,\ell)}$, $q_i^{(k,\ell)}$ and $r_i^{(k,\ell)}$ are given by:

$$p_i^{(k,\ell)} = (\sigma_i^{(k)})^2 \max \left\{0, \frac{\partial f_i}{\partial x_j}(\hat{x}^{(k)}) \right\} + \frac{\rho_i^{(k,\ell)} r_i^{(k)}}{4}$$  \hspace{1cm} (9)

$$q_i^{(k,\ell)} = (\sigma_i^{(k)})^2 \max \left\{0, -\frac{\partial f_i}{\partial x_j}(\hat{x}^{(k)}) \right\} + \frac{\rho_i^{(k,\ell)} \sigma_i^{(k)}}{4}$$  \hspace{1cm} (10)

$$r_i^{(k,\ell)} = f_i(\hat{x}^{(k)}) - \sum_{j=1}^{n} \left( \frac{p_i^{(k,\ell)} + q_i^{(k,\ell)}}{\sigma_j^{(k)}} \right).$$  \hspace{1cm} (11)

The updating of parameters $\rho_i^{(k,\ell)}$ and $\sigma_j^{(k)}$ discussed in the sequel are suggested in [8]. As far as the parameter $\rho_i^{(k,\ell)}$, for $\ell = 0$ the following values are used:

$$\rho_i^{(1,0)} = 1; \quad \rho_i^{(k+1,0)} = \max\{0.1 \rho_i^{(k,\ell)}, \rho_i^{\min}\},$$  \hspace{1cm} (12)

where $\ell(k)$ is the number of inner iterations necessary to complete the $k$-th outer iteration and $\rho_i^{\min}$ is a fixed strictly positive number.

In each inner iteration, the updating of $\rho_i^{(k,\ell)}$ is based on the solution of the latest subproblem. If $g_i^{(k,\ell)}(\hat{x}^{(k,\ell)}) < f_i(\hat{x}^{(k,\ell)})$, it is chosen $\rho_i^{(k,\ell+1)}$ such that $g_i^{(k,\ell+1)}(\hat{x}^{(k,\ell)}) = f_i(\hat{x}^{(k,\ell)})$, which gives $\rho_i^{(k,\ell+1)} = \rho_i^{(k,\ell)} + \delta_i^{(k,\ell)}$ where

$$\delta_i^{(k,\ell)} = \frac{f_i(\hat{x}^{(k,\ell)}) - g_i^{(k,\ell)}(\hat{x}^{(k,\ell)})}{w_i^{(k)}(\hat{x}^{(k,\ell)})}. \hspace{1cm} (13)$$

Thus,

$$\rho_i^{(k,\ell+1)} = \min\{10 \rho_i^{(k,\ell)}, 1.1 (\rho_i^{(k,\ell)} + \delta_i^{(k,\ell)})\} \quad \text{if} \quad \delta_i^{(k,\ell)} > 0,$$

$$\rho_i^{(k,\ell+1)} = \rho_i^{(k,\ell)} \quad \text{if} \quad \delta_i^{(k,\ell)} \leq 0. \hspace{1cm} (14)$$

It is worth noticing that in the beginning of each inner iteration, the parameters $\rho_i$ increase or remain the same, but are never reduced. Therefore, it is important that they decrease whenever an outer iteration starts, otherwise the method could generate too small steps.

In terms of the parameters $\sigma_j^{(k)}$, the updating depends on the functions $v_i^{(k)}$ and $w_i^{(k)}$. For the MMA approximations, the Hessian matrix $\nabla^2 f(x)$ is diagonal with $\frac{\partial^2 w_i^{(k)}}{\partial x_j^2}(x) \geq \frac{1}{(\sigma_j^{(k)})^2}$ for all $j$ and $x$, with equality if $x_j = x_j^{(k)}$. This means that the curvature of the function $w_i^{(k)}$ towards $x_j$ increases as $\sigma_j^{(k)}$ decreases. Thus, depending on the pattern of the variables in the previous iterations, they should be stabilized or released, according to the following rule. If $k = 1$ and $k = 2$:

$$\sigma_j^{(k)} = 0.5(x_j^{\max} - x_j^{\min}); \hspace{1cm} (15)$$

and for $k \geq 3$:

$$\sigma_j^{(k)} = \gamma_j^{(k)} \sigma_j^{(k-1)}, \hspace{1cm} (16)$$

where

$$\gamma_j^{(k)} = \begin{cases} 
0.7 & \text{if} \quad (x_j^{(k)} - x_j^{(k-1)})(x_j^{(k-1)} - x_j^{(k-2)}) < 0, \\
1.2 & \text{if} \quad (x_j^{(k)} - x_j^{(k-1)})(x_j^{(k-1)} - x_j^{(k-2)}) > 0, \\
1 & \text{if} \quad (x_j^{(k)} - x_j^{(k-1)})(x_j^{(k-1)} - x_j^{(k-2)}) = 0. 
\end{cases} \hspace{1cm} (17)$$

The structure of this manuscript is the following. In Section 4 a discussion of second order information is provided to motivate the usage of the spectral parameter, which is detailed in Section 5. The
proposed algorithms are presented in Section 6, and the numerical results are given in Section 7. Final remarks, in Section 8, conclude the text.

4. Second order information

In 1987, Svanberg [4] presented the Method of the Moving Asymptotes (MMA), that may be interpreted as a generalization of the method CONLIN, [3], in which each approximation $g_i$, denoted by $g_i^{MMA}$ is obtained by a linearization of the original functions in variables of the type $1/(x_j - l_j)$ or $1/(u_j - x_j)$, depending on the sign of the partial derivatives of $f_i$ in relation to $x^{(k)}$, where the poles of the moving asymptotes $l_j$ e $u_j$ are defined in [3]:

$$g_i^{MMA}(x) = \sum_{j=1}^{n} \left( \frac{u_j - x_j}{u_j^{(k)} - x_j} \right) \frac{\partial f_i}{\partial x_j}(x^{(k)}) - \sum_{j=1}^{n} \frac{(x_j^{(k)} - l_j^{(k)})^2}{x_j - l_j^{(k)}} + r_i^{(k)}$$

The symbols $\sum_+$ and $\sum_-$ denote the sum on the terms with positive and negative first-order derivatives, respectively. The coefficients $p_{ij}^{(k)}$, $q_{ij}^{(k)}$ are given by:

$$p_{ij}^{(k)} = \frac{(u_j^{(k)} - x_j^{(k)})^2}{x_j^{(k)} - x_j} \max \left\{ 0, \frac{\partial f_i}{\partial x_j}(x^{(k)}) \right\}$$

$$q_{ij}^{(k)} = \frac{(x_j^{(k)} - l_j^{(k)})^2}{x_j - l_j^{(k)}} \max \left\{ 0, -\frac{\partial f_i}{\partial x_j}(x^{(k)}) \right\}$$

and $r_i^{(k)}$ is given by [11], removing index $\ell$. Notice that $u_j^{(k)} - x_j^{(k)} = x_j^{(k)} - l_j^{(k)} = q_i^{(k)}$.

In the original version of MMA, in each outer iteration, given the current point $x^{(k)}$, a subproblem is generated and solved, and its solution defines the next approximation $x^{(k+1)}$, so that just a single inner iteration is performed.

The second derivative of the MMA approximation $g_i^{MMA}$ is given by:

$$\frac{\partial^2 g_i^{MMA}(x)}{\partial x_j^2} = \frac{2p_{ij}^{(k)}}{(u_j^{(k)} - x_j)^3} + \frac{2q_{ij}^{(k)}}{(x_j - l_j^{(k)})^3}$$

so that if $\frac{\partial f_i}{\partial x_j}(x^{(k)}) \geq 0$ then,

$$\frac{\partial^2 g_i^{MMA}(x)}{\partial x_j^2} = \frac{2(u_j^{(k)} - x_j)(x_j - l_j^{(k)})^2}{(u_j^{(k)} - x_j)^3},$$

and if $\frac{\partial f_i}{\partial x_j}(x^{(k)}) < 0$ then,

$$\frac{\partial^2 g_i^{MMA}(x)}{\partial x_j^2} = -\frac{2(x_j - l_j^{(k)})^2}{(x_j - l_j^{(k)})^3}.$$
After several versions of the MMA (cf. [5, 6, 7, 2]), Svanberg proposed, in 2002 [8], the MMA version already described in Section 3.

The introduction of the terms $\rho_i^{(k,\ell)}$ and $\sigma_j^{(k)}/4$ in the coefficients $p_{ij}^{(k,\ell)}$, $q_{ij}^{(k,\ell)}$, in [6] and [10], respectively, is the basic difference between the original version of MMA and its more recent one.

In the sequel a possible motivation for such terms is given, with the aim of analyzing the param-
eters $\rho_i^{(k,\ell)}$ and $\sigma_j^{(k)}$ of the method. To start with, let us consider the simple example of a quadratic approximation that rests upon such parameters:

$$g_i^Q(x) = f_i(x^{(k)}) + \nabla f_i(x^{(k)}) (x - x^{(k)}) + \frac{\rho_i^{(k,\ell)}}{2} \sum_{j=1}^n \left( \frac{x_j - x_j^{(k)}}{\sigma_j^{(k)}} \right)^2. \quad (24)$$

Since computing second order derivatives of the original functions $f_i$ is out of the question, they are approximated by the scalars $\rho_i^{(k,\ell)}/(\sigma_j^{(k)})^2$, that is,

$$\frac{\partial^2 g_i^Q(x)}{\partial x_j^2} = \frac{\rho_i^{(k,\ell)}}{\sigma_j^{(k)}}, \quad (25)$$

Back to the approximations $g_i^{(k,\ell)}$, given in [1], we have that, if $\frac{\partial f_j}{\partial x_j}(x^{(k)}) \geq 0$, then

$$\frac{\partial^2 g_j(x)}{\partial x_j^2} = \frac{2(\sigma_j^{(k)})^2 \frac{\partial f_j}{\partial x_j}(x^{(k)})}{(u_j^{(k)} - x_j)^3} + \frac{\rho_i^{(k,\ell)} \sigma_j^{(k)}}{2(u_j^{(k)} - x_j)^3} + \frac{\rho_i^{(k,\ell)} \sigma_j^{(k)}}{2(x_j - l_j^{(k)})^3}, \quad (26)$$

and, if $\frac{\partial f_j}{\partial x_j}(x^{(k)}) < 0$, then

$$\frac{\partial^2 g_j(x)}{\partial x_j^2} = -\frac{2(\sigma_j^{(k)})^2 \frac{\partial f_j}{\partial x_j}(x^{(k)})}{(x_j - l_j^{(k)})^3} + \frac{\rho_i^{(k,\ell)} \sigma_j^{(k)}}{2(u_j^{(k)} - x_j)^3} + \frac{\rho_i^{(k,\ell)} \sigma_j^{(k)}}{2(x_j - l_j^{(k)})^3}. \quad (27)$$

The first entries in each of the equalities above are equal to [21, 19], and [22, 23], respectively. The last two entries of such equalities ensure that the approximations $g_i^{(k,\ell)}$ are strictly convex, since the parameters $\rho_i^{(k,\ell)}$ and $\sigma_j^{(k)}$ are strictly positive. Moreover, they are nonmonotone, since both $x_j = l_j^{(k)}$ and $x_j = u_j^{(k)}$ are asymptotes.

Due to the separability of the approximations under consideration, let us consider next real-valued functions of a single real variable, to simplify the exposition. Given real constants $l$ and $u$ such that $l < u$, Taylor’s expansion up to the second order term in the variable $1/(x - l)$, assuming $f : \mathbb{R} \to \mathbb{R}$, is:

$$g_{22}(x) = -\frac{f’(x^{(k)})(x^{(k)} - l)^2}{x - l} + f’(x^{(k)}) (x^{(k)} - l) + f(x^{(k)})$$

$$+ \frac{f’’(x^{(k)}) (x^{(k)} - l)^3}{(x - l)^2} - \frac{2 f’’(x^{(k)}) (x^{(k)} - l)^2}{x - l} + f’’(x^{(k)}) (x^{(k)} - l)$$

$$+ \frac{f’’’(x^{(k)}) (x^{(k)} - l)^4}{2(x - l)^2} - \frac{f’’’(x^{(k)}) (x^{(k)} - l)^3}{x - l} + f’’’(x^{(k)}) (x^{(k)} - l)^2. \quad (29)$$

Likewise, in the variable $1/(u - x)$, we have:

$$g_u^{22}(x) = \frac{f’(x^{(k)}) (u - x)^2}{u - x} - f’(x^{(k)}) (u - x^{(k)}) + f(x^{(k)})$$

$$- \frac{f’’(x^{(k)}) (u - x)^3}{(u - x)^2} + \frac{2 f’’(x^{(k)}) (u - x^{(k)})^2}{u - x} - f’’(x^{(k)}) (u - x^{(k)})$$

$$+ \frac{f’’’(x^{(k)}) (u - x)^4}{2(u - x)^2} - \frac{f’’’(x^{(k)}) (u - x^{(k)})^3}{u - x} + f’’’(x^{(k)}) (u - x^{(k)})^2. \quad (32)$$

$$- \frac{f’(x^{(k)}) (u - x^{(k)})}{u - x} + f(x^{(k)})$$

$$- \frac{f’’(x^{(k)}) (u - x^{(k)})^2}{(u - x)^2} + \frac{2 f’’(x^{(k)}) (u - x^{(k)})^3}{u - x} - f’’(x^{(k)}) (u - x^{(k)})^2$$

$$+ \frac{f’’’(x^{(k)}) (u - x^{(k)})^4}{2(u - x)^2} - \frac{f’’’(x^{(k)}) (u - x^{(k)})^3}{u - x} + f’’’(x^{(k)}) (u - x^{(k)})^2. \quad (33)$$

$$5$$
The first entry of the right-hand side of (28) may be related to the coefficient \( q_{ij}^{(k)} \) in (20) and with the the first entry of coefficient \( q_{ij}^{(k,l)} \) in (10). In a similar way, the first entry of the right-hand side of (31) is related to \( p_{ij}^{(k)} \) in (19) and with the first entry of \( p_{ij}^{(k,l)} \) in (9).

The terms \( \rho_i^{(k,l)}(\sigma_i^{(k)})/4 \) present in the coefficients \( p_{ij}^{(k,l)} \) and \( q_{ij}^{(k,l)} \), in (9) and (10) respectively, might have their origin in \( g_{12} \) and \( g_{n2} \). Indeed, analyzing the second terms of (30) and of (33), if the second derivative \( f''(x^{(k)}) \) is approximated by the ratio \( \rho(x^2) \), noticing that \( (u-x^{(k)}) = (x^{(k)}-l) = \sigma \), we obtain the parameters \( \rho_i^{(k,l)} \) and \( \sigma_i^{(k)} \) of \( p_{ij}^{(k,l)} \) and \( q_{ij}^{(k,l)} \) in (9) and (10), except for the sign, that is changed to ensure convexity of the approximation. Such approximation of the second order derivatives is similar to the one used in the quadratic approximation, that is, \( \frac{\partial^2 f_i(x)}{\partial x_j^2} \approx \frac{p_i^{(k,l)}}{(\sigma_j^{(k)})^2} \) and \( f''(x^{(k)})(u-x^{(k)})^3 \) are the ones at choice because, besides carrying second order information, they have the same denominator of the approximations \( g_i^{(k,l)} \). Such relationship is exploited in the definition of the spectral parameter, which will provide a way to incorporate second order information in the updating of the parameters \( \rho_i^{(k,l)} \), as detailed in the next section.

5. The spectral parameter

The motivation for using the spectral parameter in the MMA comes from the need of avoiding computing second order derivatives of the original functions \( f_i \), \( i=0,1,\ldots,m \), due to the intrinsic involved cost. Thus, a possible way to somehow use second-order information without an excessive overload is to approximate the Hessian by cheap matrices with a simple structure.

From the Mean Value Theorem of the Integral Calculus we know that, given a continuously differentiable function \( f : \mathbb{R}^n \to \mathbb{R} \), we have that \( f(y) = f(x) + \nabla f(x + \alpha(y-x))^T(y-x) \), for some \( \alpha \in (0,1) \). Moreover, if \( f \) is twice continuously differentiable, then:

\[
\nabla f(y) = \nabla f(x) + \int_0^1 \nabla^2 f(x + \alpha(y-x)) \, d\alpha \, (y-x).
\]

By setting \( s = y-x \), the scalar

\[
\eta = \frac{s^T t}{s^T s},
\]

where \( t = \nabla f(y) - \nabla f(x) \), or again by the Mean Value Theorem, \( t = (\int_0^1 \nabla^2 f(x + \alpha s) \, d\alpha \, s) \), defines a Rayleigh quotient with respect to the average Hessian matrix \( (\int_0^1 \nabla^2 f(x + \alpha s) \, d\alpha) \). Such quotient has its value between the smallest and the largest eigenvalue of the average Hessian matrix, what motivates the terminology spectral parameter for (34). Thus, if we require that the Hessian of the functions \( f_i \) are approximated by scalar matrices, we might say that \( qI \) is the matrix of such type that best approximates the average Hessian (see [1] and references therein).

In the MMA context, the idea is to use the spectral parameter (34) in the following way:

\[
\nabla^2 f_i(x) \approx \eta_i I \quad \Rightarrow \quad \frac{\partial^2 f_i(x)}{\partial x_j^2} \approx \eta_i.
\]

Such approximation will be used in the MMA if conservativity [6] does not hold for some approximating function \( g_i^{(k,l)} \) in \( \hat{x}^{(k,l)} \). In such case, the corresponding function is modified with an increased parameter \( \rho_i \), that is, by computing \( \rho_i^{(k,l+1)} \) such that \( \rho_i^{(k,l+1)} > \rho_i^{(k,l)} \). The second order information contained in the spectral parameter is then used to obtain \( \rho_i^{(k,l+1)} \).

The points used to compute the direction \( s \) are the current estimate \( x^{(k)} \) and the solution of the latest subproblem \( \hat{x}^{(k,l)} \), that is, \( s^{(k,l)} = \hat{x}^{(k,l)} - x^{(k)} \). It is also necessary to compute vectors \( t_i^{(k,l)} = \nabla f_i(\hat{x}^{(k,l)}) - \nabla f_i(x^{(k)}) \) for the indices \( i \in \{0,1,\ldots,m\} \) associated to the approximations \( g_i^{(k,l)} \) that do not verify condition (6), so that:

\[
\rho_i^{(k,l+1)} = \rho_i^{(k,l)} + \eta_i.
\]
\[ \eta_{ij}^{(k,\ell)} = \frac{(s^{(k,\ell)})^T t_{ij}^{(k,\ell)}}{(g^{(k,\ell)})^T g^{(k,\ell)}}. \]  

(35)

It is worth noticing that to compute vectors \( t_i \), there is no need to explicitly use the variables \( x_j \), and so we will relate the second order information contained in the spectral parameter only by means of the parameter \( \rho_i \), and not with both \( \rho_i \) and \( \sigma_j \) as done by Svanberg in [3]. Due to such difference, but keeping in mind the analysis done on the expansions \( g_{12} \) and \( g_{22} \) given in (28)-(30) and (31)-(33), respectively, we will build new coefficients \( p_{ij}^{(k,\ell)} \) and \( q_{ij}^{(k,\ell)} \) as follows:

\[
 p_{ij}^{(k,\ell)} = \frac{1}{\sigma_j^{(k)}} \left[ \max \left\{ 0, \frac{\partial f_i}{\partial x_j}(x^{(k)}) + \rho_i^{(k,\ell)} \sigma_j^{(k)} \right\} \right],
\]

(36)

\[
 q_{ij}^{(k,\ell)} = \frac{1}{\sigma_j^{(k)}} \left[ \max \left\{ 0, -\frac{\partial f_i}{\partial x_j}(x^{(k)}) + \rho_i^{(k,\ell)} \sigma_j^{(k)} \right\} \right].
\]

(37)

The coefficients \( p_{ij}^{(k,\ell)} \) are computed as in (11). We have also kept the first-order approximating functions \( g_i^{(k,\ell)} \) as in (7), the definition of the subproblem set \( X^{(k)} \) given in (4), and the values of the parameters \( \sigma_i^{(k)} \) computed as in (15)-(17). As far as the parameters \( \rho_i^{(k,\ell)} \) present in (36) and (37), the value used if \( \ell = 0 \) was as in (12), with \( \rho_i^{\text{min}} = 10^{-5} \). The parameters \( \rho_i^{(k,\ell)} \) must be strictly positive. According to Svanberg [3], in each inner iteration, the updating of \( \rho_i^{(k,\ell)} \) should be based on the solution of the latest subproblem.

Writing the approximation \( g_i^{(k,\ell)} \), defined in (7), in the form \( g_i^{(k,\ell)}(x) = v_i^{(k)}(x) + \rho_i^{(k,\ell)} w_i^{(k)}(x) \), we get:

\[
 w_i^{(k)}(x) = \sum_{j=1}^{n} \frac{2(\sigma_j^{(k)})^2(x_j - x_{j})^2}{(\sigma_j^{(k)})^2 - (x_j - x_{j})^2}. \]

(38)

Svanberg’s heuristic to update the parameters \( \rho_i^{(k,\ell)} \) is also based on the values \( \delta_i^{(k,\ell)} \), given in (13), that provide a relative correction so that the conservativity condition could be satisfied by the approximation \( g_i^{(k,\ell)} \). To obtain global convergence of the MMA, the choices of Svanberg were as in (14).

To update the parameters \( \rho_i \) between two consecutive inner iterations we adapt the approach given in [14]. To update \( \rho_i^{(k,\ell)} \) for the indices \( i = 0, 1, \ldots, m \) associated to the approximations \( g_i^{(k,\ell)} \) for which conservativity in \( x^{(k)} \) does not hold we proceed as follows:

1. If \( 1.1(\rho_i^{(k,\ell)} + \delta_i^{(k,\ell)}) \leq 10\rho_i^{(k,\ell)} \) then set \( \rho_i^{(k,\ell+1)} = 1.1(\rho_i^{(k,\ell)} + \delta_i^{(k,\ell)}) \).

2. Otherwise, if \( \rho_i^{(k,\ell)} < \eta_i^{(k,\ell)} < 1.1(\rho_i^{(k,\ell)} + \delta_i^{(k,\ell)}) \) then set \( \rho_i^{(k,\ell+1)} = K_1(\eta_i^{(k,\ell)}) + K_2(1.1(\rho_i^{(k,\ell)} + \delta_i^{(k,\ell)})) \), with \( K_1 + K_2 = 1 \), \( K_1, K_2 \geq 0 \).

3. Now, if \( \eta_i^{(k,\ell)} > 1.1(\rho_i^{(k,\ell)} + \delta_i^{(k,\ell)}) \) then set \( \rho_i^{(k,\ell+1)} = K_3(\eta_i^{(k,\ell)}) + K_4(1.1(\rho_i^{(k,\ell)} + \delta_i^{(k,\ell)})) \), with \( K_3 + K_4 = 1 \), \( K_3, K_4 \geq 0 \).

4. Finally, if \( \eta_i^{(k,\ell)} \leq \rho_i^{(k,\ell)} \) then set \( \rho_i^{(k,\ell+1)} = 10\rho_i^{(k,\ell)} \).

6. The algorithms
The algorithm that follows describes the whole process, for a given initial estimate \( x^{(1)} \) ∈ \( X \).

**Algorithm 1: Method of Moving Asymptotes with Spectral Updating**

**Step 1. Initialization**
Define \( \rho_i^{(1)} \) as in (12) for \( i = 0, 1, \ldots, m \). Set \( k = 1 \).

**Step 2. Stopping criteria**
If \( x^{(k)} \) satisfies the KKT conditions of problem (2), stop and take \( x^{(k)} \) as a solution.

**Step 3. Compute the parameters \( \rho_i^{(k,\ell)} \) and \( \sigma_j^{(k)} \) and the poles of the asymptotes \( t_i^{(k)} \) and \( w_j^{(k)} \)
Set $\ell = 0$.
If $k > 1$, compute $\rho_i^{(k, \ell)}$ for $i = 0, 1, \ldots, m$ as in (12).
Compute $\sigma_j^{(k)}$ for $j = 1, \ldots, n$ as in (15)–(17).
Compute $\rho_j^{(k)} = \sigma_j^{(k)}$ and $w_j^{(k)} = x_j^{(k)} + \sigma_j^{(k)}$, for $j = 1, \ldots, n$. 

Step 4. Generates and solves the subproblem
Compute the coefficients $p_{ij}^{(k, \ell)}$ and $q_{ij}^{(k, \ell)}$, for $i = 0, 1, \ldots, m$ and $j = 1, \ldots, n$ as in (39), (37) and (11), respectively.
Defines the approximations $g_i^{(k, \ell)}$, for $i = 0, 1, \ldots, m$ as in (9).
Solves the subproblem, obtaining $\hat{x}^{(k, \ell)}$.

Step 5. Tests the conservativity condition
If (6) does not hold for some index $i = 0, 1, \ldots, m$ 

Step 5.1. Updating of parameters $\rho_i^{(k, \ell)}$.

Step 5.2. Start a new inner iteration.
Set $\ell = \ell + 1$. Go to Step 4.
Otherwise, set $x^{(k+1)} = \hat{x}^{(k, \ell)}$, $k = k + 1$ and go to Step 2.

The updating of the parameters $\rho_i^{(k, \ell)}$ (Step 5.1) is described in the next algorithm.

Algorithm 2: Step 5.1 (Updating of parameters $\rho_i^{(k, \ell)}$)
Compute $s^{(k, \ell)} = \hat{x}^{(k, \ell)} - x^{(k)}$, and, for the indices $i = 0, 1, \ldots, m$ associated to the approximations $g_i^{(k, \ell)}$ for which conservativity does not hold in $\hat{x}^{(k, \ell)}$, compute

$$
\begin{align*}
\hat{t}_i^{(k, \ell)} &= \nabla f_i(\hat{x}^{(k, \ell)}) - \nabla f_i(x^{(k)}); \\
\eta_i^{(k, \ell)} &= \frac{s^{(k, \ell)}Tt_i^{(k, \ell)}}{s^{(k, \ell)}Ts^{(k, \ell)}}; \\
\delta_i^{(k, \ell)} &= \frac{f_i(\hat{x}^{(k, \ell)}) - g_i^{(k, \ell)}(\hat{x}^{(k, \ell)})}{w_i^{(k)}(\hat{x}^{(k, \ell)})}.
\end{align*}
$$

If $1.1(\rho_i^{(k, \ell)} + \delta_i^{(k, \ell)}) \leq 10\rho_i^{(k, \ell)}$ set $\rho_i^{(k, \ell+1)} = 1.1(\rho_i^{(k, \ell)} + \delta_i^{(k, \ell)})$;
Otherwise
If $\eta_i^{(k, \ell)} > \rho_i^{(k, \ell)}$
If $\eta_i^{(k, \ell)} \leq 1.1(\rho_i^{(k, \ell)} + \delta_i^{(k, \ell)})$ set $\rho_i^{(k, \ell+1)} = K_1(\eta_i^{(k, \ell)}) + K_2(1.1(\rho_i^{(k, \ell)} + \delta_i^{(k, \ell)}))$;
Otherwise, set $\rho_i^{(k, \ell+1)} = K_3(\eta_i^{(k, \ell)}) + K_4(1.1(\rho_i^{(k, \ell)} + \delta_i^{(k, \ell)}))$;
End
Otherwise, $\rho_i^{(k, \ell+1)} = 10\rho_i^{(k, \ell)}$;
End

It is worth noticing that the values $\delta_i^{(k, \ell)}$ of Algorithm 2 are computed only for the indices $i = 0, 1, \ldots, m$ associated to the approximations $g_i^{(k, \ell)}$ for which conservativity does not hold. Observing that the function $w_i^{(k)}$ given in (38) is always positive, the values $\delta_i^{(k, \ell)}$ are also always positive as well.

As far as the values for the constants $K_1$, $K_2$, $K_3$ and $K_4$, whenever $\eta_i^{(k, \ell)} > 1.1(\rho_i^{(k, \ell)} + \delta_i^{(k, \ell)})$, the weight given to the parameter $\eta_i^{(k, \ell)}$ through the constant $K_3$ cannot be large because such parameter may reach quite large values. Reasonable choices are $K_3 = 0.1$ and $K_4 = 0.9$. Now, if $\eta_i^{(k, \ell)} \leq 1,1(\rho_i^{(k, \ell)} + \delta_i^{(k, \ell)})$, more weight can be given to parameter $\eta_i^{(k, \ell)}$ setting, for instance, $K_1 = K_2 = 0.5$.

7. Numerical results
This section contains the description of the computational tests performed by applying the MMA with the spectral updating, described in Algorithms 1 and 2, in two academic problems with variable dimension, proposed in [8]. The experiments were run in Matlab, version 7.0.0.19920(R14), in a Pentium D, 3.4GHz, 2 GB memory RAM, OS Windows XP.

The general structure of the test problems is similar to the one in topology optimization: non-convex problems with a large number of variables, lower and upper bounds on every variable and a small number of inequality constraints.
Three symmetric positive definite matrices $S$, $P$ and $Q$, $n \times n$, are used in both problems, the elements of which are given by:

$$s_{ij} = \frac{2 + \sin(4\pi \alpha_{ij})}{(1 + |i - j|) \ln n}, \quad p_{ij} = \frac{1 + 2\alpha_{ij}}{(1 + |i - j|) \ln n}, \quad q_{ij} = \frac{3 - 2\alpha_{ij}}{(1 + |i - j|) \ln n},$$

where $\alpha_{ij} = \frac{i + j - 2}{2n - 2} \in [0, 1]$ for all $i$ and $j$.

In the first problem, the objective function is strictly convex, but the constraints are strictly concave, so that the feasible set is nonconvex. Its formulation is given next, and the initial estimate is $x^{(i)} = (0.5, 0.5, \ldots, 0.5)^T \in \mathbb{R}^n$.

**Problem 1:**

$$\begin{align*}
\text{minimize} & \quad f_0(x) = x^T S x \\
\text{subject to} & \quad f_1(x) = n/2 - x^T P x \leq 0, \\
& \quad f_2(x) = n/2 - x^T Q x \leq 0, \\
& \quad -1 \leq x_j \leq 1, \quad j = 1, \ldots, n. 
\end{align*}$$  \hspace{1cm} (39)

In the second problem, the constraints are strictly convex, but the objective function is strictly concave. It is stated below, and the initial estimate is $x^{(i)} = (0.25, 0.25, \ldots, 0.25)^T \in \mathbb{R}^n$.

**Problem 2:**

$$\begin{align*}
\text{minimize} & \quad f_0(x) = -x^T S x \\
\text{subject to} & \quad f_1(x) = x^T P x - n/2 \leq 0, \\
& \quad f_2(x) = x^T Q x - n/2 \leq 0, \\
& \quad -1 \leq x_j \leq 1, \quad j = 1, \ldots, n. 
\end{align*}$$  \hspace{1cm} (40)

For both problems, four dimensions were considered: $n = 100, 500, 1000$ and $2000$. It is worth noticing that the problems are formulated as (1), so initially they were posed in the format (2) with $a_0 = 1, a_1 = a_2 = 0, d_1 = d_2 = 1$ and $c_1 = c_2 = 1000$. These choices provided $y \equiv 0$ an $z \equiv 0$ in each outer iterate.

The subproblems were solved using a primal-dual interior point method, as suggested in [6]. As far as the stopping criterion, the outer iterations finished whenever the square of the norm of the residue of the KKT conditions of the enlarged problem gets less than or equal to $10^{-10}$.

The results concerning problems (39) and (40) are given in Tables 1 and 2, respectively. For each problem, three instances were analyzed. Instance 1 corresponds to the MMA method of Svanberg, implemented exactly as in [3]. For instance 2, the coefficients (36) and (37) were used to assemble the approximations $g_i^{(k, \ell)}$, the function (38) was used to build the $\delta_i^{(k, \ell)}$, but the strategy (14) was kept to update the parameter $\rho_i^{(k, \ell)}$ between two consecutive inner iterations. Finally, instance 3 was performed according to Algorithms 1 and 2, that is, the coefficients (36) and (37) were used to assemble the approximations $g_i^{(k, \ell)}$, and the information provided by the spectral parameter was used to update the parameter $\rho_i^{(k, \ell)}$ between two consecutive inner iterations. In each table, the first column contains the number of variables $n$, and the second, the objective function value at the final point. In the third, fourth and fifth columns, triples are given for each instance, with the corresponding number of outer iterations performed, the number of additional inner iterations, and the CPU time (in seconds) (among round brackets). The total number of solved subproblems is the sum of the outer with the additional inner iterations.

**Table 1:** Numerical results for Problem 1.

<table>
<thead>
<tr>
<th>Number of variables</th>
<th>Objective function value</th>
<th>Instance 1 (Svanberg)</th>
<th>Instance 2 (modified Svanberg)</th>
<th>Instance 3 (spectral updating)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>24.90</td>
<td>104 (135); 15.76</td>
<td>101 (117); 14.72</td>
<td>102 (104); 15.76</td>
</tr>
<tr>
<td>500</td>
<td>129.65</td>
<td>147 (185); 388.37</td>
<td>154 (173); 388.34</td>
<td>154 (159); 433.55</td>
</tr>
<tr>
<td>1000</td>
<td>260.85</td>
<td>174 (222); 1783.97</td>
<td>185 (180); 1774.03</td>
<td>186 (178); 1983.34</td>
</tr>
<tr>
<td>2000</td>
<td>523.51</td>
<td>185 (229); 6502.53</td>
<td>195 (209); 7487.02</td>
<td>200 (199); 7380.22</td>
</tr>
</tbody>
</table>

To perform instance 3, the constants $K_1, K_2, K_3$ and $K_4$ of Algorithm 2 were chosen as $K_1 = 0.2, K_2 = 0.8, K_3 = 0.025$ and $K_4 = 0.975$. It is worth mentioning that the constants $K_1, K_2$ were never
Table 2: Numerical results for Problem 2.

<table>
<thead>
<tr>
<th>Number of variables</th>
<th>Objective function value</th>
<th>Instance 1 (Svanberg)</th>
<th>Instance 2 (modified Svanberg)</th>
<th>Instance 3 (spectral updating)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>-75.10</td>
<td>218 (265); 36.78</td>
<td>223 (250); 32.31</td>
<td>227 (240); 37.03</td>
</tr>
<tr>
<td>500</td>
<td>-370.35</td>
<td>392 (415); 1002.55</td>
<td>389 (398); 961.30</td>
<td>394 (388); 1163.69</td>
</tr>
<tr>
<td>1000</td>
<td>-739.15</td>
<td>438 (437); 4395.86</td>
<td>441 (418); 4229.44</td>
<td>449 (408); 5033.81</td>
</tr>
<tr>
<td>2000</td>
<td>-1476.49</td>
<td>478 (501); 16268.09</td>
<td>477 (481); 18073.42</td>
<td>491 (481); 18896.39</td>
</tr>
</tbody>
</table>

used, since when $1.1(\rho^{(k,\ell)} + \delta^{(k,\ell)}) > 10\rho^{(k,\ell)}$, the case $\rho^{(k,\ell)} < \eta^{(k,\ell)}(k,\ell)$ never occurred. As far as the constants $K_3 \in K_4$, the weight given to the spectral parameter, through $K_3$, is relatively small to avoid a too conservative approximation, and consequently, too small steps.

Analyzing the results of Tables 1 and 2, we can notice that in most of the cases, instance 3 performs more outer iterations to reach convergence than the other two instances. However, in instance 3, the total number of solved subproblems is smaller than the one of instance 1. Indeed, in the tests with instance 3, on average, each outer iteration requires one additional inner iteration, whereas with instances 1 and 2 the average is slightly greater than one, being this difference more significant for Problem 1 than it is for Problem 2.

In terms of the CPU time, computing the time spent by each inner iteration in the three instances, we observe that the extra evaluation of the gradient for the functions for which conservativity was violated, required by instance 3, does not overload the demanded effort for both problems. Considering the tests with $n = 2000$, for example, for instances 1, 2 and 3, these values are 15.7, 18.5 and 18.5 seconds for Problem 1 and 16.6, 18.7 and 19.4 seconds for Problem 2.

8. Final remarks
Although the obtained results in the numerical experiments were satisfactory, we would like to improve them with alternative approaches for initializing the parameter $\rho$ and for the updating of this parameter in the beginning of each outer iteration. We also intend to somehow relax conservativity, and investigate a strategy to solve the subproblems that combines the dual approach with trust region ideas, developing implementations of the algorithms in Fortran.

9. References